

Designing social networks for efficient learning

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May 10, 2016

Abstract

We study structural properties for information aggregation in settings where self-interested agents take actions sequentially and partially observe each other. Agents optimize their choice of action based on some local information (or ‘private signal’) as well as conclusions they draw from observing actions taken by others. The literature so far typically studies the case where all agents observe each other. In such a setting it has been shown that information need not be aggregated and agents run a risk of herding on an inferior action. The other extreme, where agents do not observe each other at all clearly will not allow for social learning. In this paper we study the interim case where agents partly observe each other. We model partial observability by overlaying a graph structure over the agents. We consider the challenge of designing such a network structure that guarantees social learning. We introduce the ‘celebrity graph’ and prove that it induces social learning and an optimal outcome with overwhelming probability. We study variations of this setting which depend on our assumption over the order or arrival of the agents (Bayesian and adversarial).

1 Introduction

A classical setting where distributed systems such as decentralized markets may fail is when critical information exists but is distributed among the agents. Each agent can access the information held by other agents only implicitly by observing their actions. Information may then be aggregated by a market correctly, typically resulting in efficient market outcomes, or may be aggregated incorrectly (or partially) which often results in a system failure.

Consider the classical setting of sequential information aggregation from the economics literature known as the *herding* model [6]. In this setting agents arrive one after the other,

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possibly in some random order, and each has to choose some action. The value of each action is the same for all agents and depends on some underlying unknown state of nature. Each agent privately receives a stochastic signal that is correlated with the state of nature. Thus, the choice of action for the agent arriving at stage t is based on his signal as well as actions chosen (and not signals observed) by his predecessors. For the sake of a concrete setting, consider a set of tourists who choose a restaurant for dinner. The tourists may not know which place is better, but before their trip they may have gotten a signal in the form of some restaurant review: The Israeli tourist might have read a review in some local paper in Tel-Aviv, whereas the German tourist might have read some blog written by a foodie in Berlin. Each tourist also sees the choice made by some of the other tourists, those that had dinner earlier. Based on this information he must make up his mind.

As agents only share information passively it is no wonder they may not necessarily learn to take the optimal action but may ‘herd’ on some inferior action. To see this consider the case where the first few agents (possibly just two in standard examples) arriving for dinner are unlucky and receive a signal which suggests that the inferior restaurant is actually the better one. Consequently, they all go to the worse place. The next agent then sees her predecessors all in one place and will conclude (via Bayesian update) that it must be the better of the two with high probability, no matter what her private signal is. Consequently, she will follow the crowd, and her private information will be lost forever, i.e., not to be enjoyed by future agents. The agent next in line would then face the same information setting when she shows up, similarly choose to ignore her signal, and dine at the inferior restaurant. Thus, except for the initial set of agents, information is not aggregated and agents ‘herd’ on the inferior option resulting in market failure.

The primary observation in the literature is that herding can happen when signals are bounded and will not happen when signals are unbounded ([10]). Signals are called bounded when the ex-post probability of the true state of nature given the signal is bounded away from 1, across all states and all signals. Our model will consider a finite set of signals and hence the signals are bounded.

Now, consider the case where agents do not necessarily observe all predecessors but rather only those that are their friends in some given social network. That is, agents arrive in some random order and each agent observes the actions of his neighbors in the network who have arrived earlier. Each agent takes the optimal action given the information available to him. Possibly, if agents do not see too many predecessors, the herding scenario described above will not materialize (or will materialize with a vanishing probability). Thus, if one would like to design a network in order to avoid herding then a network with low connectivity seems appealing. However, on the other hand one must be careful not to have a network that is too loosely connected. In such networks agents might not have access to actions contributed by predecessors, which again might lead to market failure (to be convinced of this consider the empty network when no one sees each other).

One possible design is to hop between the two extremes - an empty network and a clique. Consider the case where an initial set of agents who cannot observe each other (let us nickname these agents as ‘guinea-pigs’) is followed by agents who see everyone (the ‘followers’). If the set of guinea-pigs is large enough then most likely the majority will take

the optimal action (this is a consequence of the law of large numbers) and so the followers will all take the optimal action as well. The problem with this design is that when agents arrive randomly one does not know in advance who the guinea-pigs should be and so the design must depend on the realized order of arrival (an ‘on-line’ design).

In this work we pursue the challenge of an off-line design of a social network that aggregates information (almost) optimally. Our main result provides a graph structure, which we nickname the *celebrities’ graph* that allows for an approximately optimal outcome. This graph implements the idea of guinea-pigs and observant followers even when the order of arrival is random. The idea behind our construction is to assign a small proportion of the population as ‘celebrities’ (this assignment, as in reality, may be independent of any personal traits or local information) and look at the bi-clique with celebrities on one side and non-celebrities on the other side.¹ It turns out that by properly choosing the size of the celebrity set and its members (these may depend on the distribution over the order at which agents arrive) one can expect a large set of non-celebrities to arrive first (and become de-facto guinea-pigs), followed by a celebrity who will see them all and follow their majority action. With our construction we can show than anyone arriving after the first celebrity will also choose the same action.

Theorem 1.1 (Informal). *The celebrities graph is asymptotically optimal in the sense that as the size of the population grows, the expected fraction of agents that take the optimal action grows to 1.*

We show that this construction holds when the arrival order is uniformly chosen from the set of all permutations. We then extend this to arbitrary distributions and to the adversarial case where the order of arrival may depend on the actual graph (in which case we resort to a random celebrity graph).

As argued above the literature on social learning and herding is almost entirely about full visibility (or no-visibility) graphs. One paper that stands out is Acemoglu et al [1]. In that paper the authors study the network properties for which information is successfully aggregated. The social network in their work is generated online as agents arrive, in a stochastic manner. That is each arriving agent gets to see a random subset of his predecessors. The random process by which such a subset is generated is critical for determining whether learning occurs. Much of their work is about unbounded signals in which case they show that for societies to aggregate information it is required that the probability that each individual observes some other individual from the recent past converges to one as the social network becomes large. In the context of our challenge the unbounded case is trivial as the full graph (clique) is known to aggregate information ([10]).

However, they also consider the bounded signal scenario where the full graph is known to fail. A necessary condition which guarantees social learning in this case is that once in a while an agent gets to observe only a set of predecessors (what they refer to as ‘non persuasive neighborhoods’) where in any possible set of actions among that predecessors the conditional probability for each possible world state is bounded, and therefore even though the agents private belief is bounded he may choose to follow his own signal.

¹A bi-clique over two disjoint sets is the maximal bi-partite graph over these two sets.

Unfortunately, the results in [1] shed no light on the design of optimal network when the order of arrival is random which is our primary concern.

While our work is among the first to introduce the impact of the social visibility network in the context of herding in a multi-agent system, network effects among agents in a multi-agent system has already played a major role in the AI and multi-agent systems literature. That literature typically models utility effects (a.k.a externalities) [7, 5, 3], while recent work has also considered information propagation among neighbors [4]. Our work also complements work on sequential voting with complete information in the CS literature [11, 8]. A more general discussion of the herding topic, and many related pointers can be found in [9].

2 Model

A social learning challenge \mathcal{L} , is a tuple $\mathcal{L} = (\Theta, P, \sigma, N, A, S, G, U)$ composed of:

- Θ is finite set of states of nature.
- $P \in \Delta(\Theta)$ is a prior distribution over the state space. We assume throughout that P has full support.
- σ is a random permutation of a set of N agents, for any N .
- N is a finite set of agents (at the risk of some confusion we often use N to refer to the cardinality of this set as well).
- A is a finite set of actions available to each agent.
- S is a finite set of signals
- $G = (G_\theta)_{\theta \in \Theta}$ where $G_\theta \in \Delta(S)$ is a probability distribution over the set of signals, S corresponding to the state θ . That is, conditional on the state θ all agents receive an independent noisy signal according to G_θ .
- $U : \Theta \times A \rightarrow R$ is the common utility function for all agents

A state of nature in $\theta \in \Theta$ is chosen randomly from a known prior distribution with full support. In a society of N agents, each receives some signal $s \in S$ drawn from the probability distribution G_θ , independently from signals received by other agents. The agents arrive according to the random permutation σ and must choose an action in A .

A social network for the set of agents, N , is a set of edges $E \subset N \times N$ where an undirected edge $(i, j) \in E$ implies that agent i can see the action chosen by agent j whenever j arrived before i (formally, $\sigma(j) < \sigma(i)$) and vice versa. Thus, when i arrives he knows who his predecessors are but may only take into account the action chosen by those predecessors that are his neighbors on the network.

A social network is called ϵ -optimal if the expected number of agents, within the set N , who take the optimal strategy in equilibrium is at least $(1 - \epsilon)N$. The question we pursue is about the feasibility of designing ϵ -optimal social networks and their structure.

To demonstrate the subtlety of the question let us consider two extreme network designs, where $E = N \times N$ or $E = \emptyset$. We argue that both are not ϵ -optimal for small enough ϵ . The fact that $E = N \times N$ is not optimal is a direct consequence of informational cascades and potential herding when signals are bounded (recall there are finitely many signals), whereas the case for $E = \emptyset$ is straightforward.

An intuitive approach may be a hybrid of the two aforementioned approaches - to allow some agents to choose actions independently of each other (that is not to allow them to observe each other) and once sufficient observations have been made the optimal action becomes quite significant. We refer to this set of agents as the ‘guinea-pigs’. Following the set of guinea-pigs we allow all subsequent agents to observe all agents. This approach also fails as the order at which the agents arrive is random and so one cannot designate the set of guinea-pigs a-priori.

In this paper we focus on the following fundamental social learning challenge where $A = S = \Theta = \{0, 1\}$, $0 < \delta \leq 0.5$, $G_\theta(\theta) = 0.5 + \delta$, $U(\theta, \theta) = 1$ and $U(\theta, 1 - \theta) = 0$ for all $\theta \in \{0, 1\}$. Our results all discuss the design of ϵ -optimal social networks for sufficiently large societies under different assumptions on the (random) order of arrival : (1) the agents arrive according to a permutation that is uniformly chosen among all permutations, (2) agents arrive according to some order that is set by an adversary who knows the graph structure, and (3) agents arrive according to a some arbitrary, but non-adversarial, random order.

The basic network structure we propose will also do well when we extend the model beyond the binary and beyond the symmetric case, although the exact size of the required population, N , as a function of ϵ , the required proximity to optimality, may change.

3 Results

In this section we propose a network design that is ϵ -optimal for a sufficiently large population. We begin with the easy case where agents’ order of arrival is deterministic and known to the designer.

3.1 Deterministic arrival order

An important building block for our analysis is to understand how to design an asymptotically optimal network for the deterministic order of arrival case. This is clearly based on the aforementioned guinea-pig approach. Consider a society with N guinea-pigs and where agents $N + 1$ observes all the guinea-pigs. Let a_n denote the action of agent n . Then:

Observation 3.1. *If all agents are playing their best-reply then $\sum_{n=1}^N a_n \geq \frac{N}{2} + 1 \implies a_{N+1} = 1$, $\sum_{n=1}^N a_n \leq \frac{N}{2} - 1 \implies a_{N+1} = 0$ and otherwise $a_{N+1} = s_{N+1}$.*

In words, whenever there is a clear majority for one action among the N guinea-pigs, agent $N + 1$ ignores his own signal and follows the majority.

We sketch the proof of this lemma which quite is straightforward: Note that a guinea-pig takes action θ whenever his signal is equal θ . Therefore the condition $\sum_{n=1}^N a_n \geq \frac{N}{2} + 1$ is the same as $\sum_{n=1}^N s_n \geq \frac{N}{2} + 1$. This implies that $\sum_{n=1}^{N+1} s_n > \frac{N+1}{2}$, no matter what the value of s_{N+1} is. This means that the majority of signals among agents $1, \dots, N + 1$ is 1. Knowing this, the optimal action for agent $N + 1$ is to take action 1. The second case follows symmetric arguments. As for the third case note that the condition implies that s_{N+1} equals the (weak) majority signal among agents $1, \dots, N + 1$ and the conclusion follows.

Lemma 3.2. *Assume $E = \emptyset$ (all agents are guinea-pigs) and let X denote the number of agents that take the correct action. Whenever $N \geq \frac{1}{4\delta^2\epsilon}$, $\text{Prob}(X \geq \frac{N+1}{2}) > 1 - \epsilon$.*

Proof: Note that as $E = \emptyset$ the only information that agents have is their signal. Maximizing expected utility implies that each agent takes the action equal the signal received. Let X be the random variable that counts the number of agents whose signal equals the state of nature and note that X has a binomial distribution with parameters $(N, 0.5 + \delta)$ and so $E(X) = N(0.5 + \delta)$ and $\text{Var}(X) = N(0.5 + \delta)(0.5 - \delta)$.

$$\begin{aligned}
\text{Prob}(X < \frac{N}{2} + 1) &= \text{Prob}(E(X) - X > E(X) - (\frac{N}{2} + 1)) \leq \\
&\leq \text{Prob}(E(X) - X > E(X) - (\frac{N}{2} + 1)) + \\
&\quad \text{Prob}(X - E(X) > E(X) - (\frac{N}{2} + 1)) = \\
&= \text{Prob}(|E(X) - X| > E(X) - (\frac{N}{2} + 1)). \tag{1}
\end{aligned}$$

Applying Chebyshev inequality:

$$\begin{aligned}
\text{Prob}(|E(X) - X| > E(X) - (\frac{N}{2} + 1)) &\leq \frac{\text{Var}(X)}{(\frac{N}{2} - E(X))^2} = \\
&= \frac{N(0.5 - \delta)(0.5 + \delta)}{(\frac{N}{2} - N(0.5 + \delta))^2} = \frac{(0.5 - \delta)(0.5 + \delta)}{N(\delta)^2}. \tag{2}
\end{aligned}$$

For $N \geq \frac{1}{4\delta^2\epsilon}$ we get $\frac{4(0.5-\delta)(0.5+\delta)}{N(2\delta)^2} \leq \epsilon$, which, together with inequalities 3.1 and 3.1 prove the result.

Q.E.D

The following corollary is immediate:

Corollary 3.3. *For any $\epsilon > 0$ there exists some integer \hat{N} such that for any society with more than \hat{N} agents and any deterministic arrival order there exists an ϵ -optimal guinea-pig social network.*

Proof: Let $\hat{N} = \frac{1}{\delta^2 \epsilon^2}$ and let the first $K = \frac{1}{4\delta^2 \frac{\epsilon}{2}}$ coming agents be the guinea-pigs. Hence, from Lemma 3.2 we get that with probability greater than $(1 - \frac{\epsilon}{2})$ a clear majority of the guinea-pigs which take to optimal action, θ . By Observation 3.1 we conclude that when such a clear majority emerges all the remaining agents also take the action θ regardless of their own signal. Denoting by X the number of agents which take action θ we get:

$$\frac{E(X)}{N} > \frac{(N - K)(1 - \frac{\epsilon}{2})}{N} \geq 1 - \epsilon$$

where the last inequality follows from our choice of \hat{N} and K .

Q.E.D

We now turn to our three main results, that differ with respect to the order in which agents arrive. We begin with the assumption that agents arrive according to a permutation that is randomly and uniformly selected among all permutations.

3.2 Random and uniform arrival order

In this subsection we focus on the random and uniform order of arrival, σ_U , where the permutation $\sigma_U : N \rightarrow N$ is drawn uniformly from the set of all permutations over N , for all N . The idea we pursue is the following - designate agents $1, \dots, K$ as ‘celebrities’ (any subset containing K agents would have worked as well). Let the agents in the complementary set be known as ‘commoners’. The ‘celebrities social network’ is a bi-clique with the K celebrities on one side and the $N - K$ commoners on the other side.

We now turn to show that for a random uniform arrival order all celebrities are likely to be absent from the set of agents that arrive initially. First we introduce the following notation - for any $M \subset N$ and a permutation σ let $\sigma(M) = \{\sigma(m) : m \in M\}$.

Lemma 3.4. *Fix K and J and let N be large enough to satisfy $N \geq \frac{2JK}{\epsilon} + J$. Then any set of K agents satisfies*

$$Prob(\{\sigma_U\{1, \dots, J\} \cap K = \emptyset\}) \geq 1 - \epsilon.$$

Proof: $Prob(\{\sigma_U\{1, \dots, J\} \cap K = \emptyset\}) = Prob(\sigma_U(1) \cap K = \emptyset) Prob(\sigma_U(2) \cap K = \emptyset | \sigma_U(1) \cap K = \emptyset) \dots Prob(\sigma_U(J) \cap K = \emptyset | (\{\sigma_U\{1, \dots, J-1\} \cap K = \emptyset)) = \frac{N-K}{N} \frac{N-K-1}{N-1} \dots \frac{N-K-J+1}{N-J+1} > (\frac{N-K-J}{N-J})^J$. Therefore it is sufficient to show that $(\frac{N-K-J}{N-J})^J \geq 1 - \epsilon$. However, this inequality is equivalent to showing that $N - J \geq \frac{K}{1-(1-\epsilon)^{\frac{1}{J}}}$. Recalling that $N - J \geq \frac{2Jk}{\epsilon}$ it suffices to show that $\frac{2Jk}{\epsilon} \geq \frac{K}{1-(1-\epsilon)^{\frac{1}{J}}}$.

It is straightforward to verify that the following inequality holds: $e^{-2} \leq (1 - \frac{1}{x})^x \leq e^{-1}$ for all $x \geq 2$.² By substituting $\frac{2J}{\epsilon}$ for x (note that indeed $\frac{2J}{\epsilon} \geq 2$) we have that $e^{-\epsilon} \leq (1 - \frac{\epsilon}{2J})^J$. As $1 - \epsilon \leq e^{-\epsilon}$ we conclude that $1 - \epsilon \leq (1 - \frac{\epsilon}{2J})^J$ or equivalently $(1 - \epsilon)^{\frac{1}{J}} \leq (1 - \frac{\epsilon}{2J})$,

²The proof of this inequality follows from three simple observations: (1) $\lim_{x \rightarrow \infty} (1 - \frac{1}{x})^x = e^{-1}$, (2) $e^{-2} \leq (1 - \frac{1}{2})^2$; and (3) $(1 - \frac{1}{x})^x$ is an increasing function for $x \geq 2$.

which in turn implies that $\frac{K}{1-(1-\epsilon)^{\frac{1}{J}}} \leq \frac{K}{1-(1-\frac{\epsilon}{2J})}$. By simple manipulations this implies that $\frac{2JK}{\epsilon} \leq \frac{K}{1-(1-\epsilon)^{\frac{1}{J}}}$ as required.

Q.E.D

Note that whenever J is of the order of magnitude of $N\epsilon$ the conditions of Lemma 3.4 are violated. Indeed it is quite likely that at least one celebrity will arrive among the first $N\epsilon$ agents, even when such celebrities are absent from the first J agents:

Lemma 3.5. *For an arbitrary $\epsilon > 0$ let K, J and N satisfy $K \geq \frac{2}{\epsilon} \ln(\frac{1}{\epsilon})$ and $N \geq \frac{2J}{\epsilon}$. Then: $Prob(\sigma_U\{J+1, \dots, N\epsilon\} \cap K \neq \emptyset | \sigma_U\{1, \dots, J\} \cap K = \emptyset) \geq 1 - \epsilon$.*

Proof:

$$\begin{aligned} & Prob(\sigma_U\{J+1, \dots, N\epsilon\} \cap K \neq \emptyset | \sigma_U\{1, \dots, J\} \cap K = \emptyset) = \\ & 1 - Prob(\sigma_U\{J+1, \dots, N\epsilon\} \cap K = \emptyset | \sigma_U\{1, \dots, J\} \cap K = \emptyset) = \\ & = 1 - \frac{N-J-K}{N-J} \frac{N-J-K-1}{N-J-1} \dots \frac{N-J-K-(N\epsilon-J)+1}{N-J-(N\epsilon-J)+1} \geq \\ & 1 - \left(\frac{N-J-K}{N-J}\right)^{N\epsilon-J}. \end{aligned}$$

Therefore it is sufficient to show $\left(\frac{N-J-K}{N-J}\right)^{N\epsilon-J} \leq \epsilon$. As $N \geq \frac{2J}{\epsilon}$ it is enough to show that $\left(\frac{N-J-K}{N-J}\right)^{\frac{N\epsilon}{2}} \leq \epsilon$.

Once again we resort to the inequality $(1 - \frac{1}{x})^x \leq e^{-1}$, $\forall x \geq 2$ and apply it to $x = \frac{N-J}{K}$ (note that indeed $\frac{N-J}{K} \geq 2$). Thus, $\left(\frac{N-J-K}{N-J}\right)^{\frac{N\epsilon}{2}} \leq e^{-1 \frac{N\epsilon}{2} \frac{K}{N-J}} \leq \epsilon$, where the last inequality follows from our assumption that $K \geq \frac{2}{\epsilon} \ln(\frac{1}{\epsilon})$.

Q.E.D

Given $\epsilon > 0$ and a set of N agents let \mathcal{H} be the bi-clique with $K = \frac{8}{\epsilon} \ln \frac{4}{\epsilon}$ celebrities and $N - K$ commoners (assume N is large enough so that this is well defined).

Lemma 3.6. *For $J \geq \frac{1}{\delta^2 \epsilon}$ and $\epsilon \leq 0.5 - \delta$. Assuming agents $1, \dots, J$ are commoners, agent $J+1$ is celebrity and agent $J+2$ is commoner and assume all agents are playing their best-reply then $\sum_{h=1}^J a_h \geq \frac{J}{2} + 1 \implies a_{J+2} = 1$ and $\sum_{h=1}^J a_h \leq \frac{J}{2} - 1 \implies a_{J+2} = 0$.*

Proof:

We have already seen (observation 3.1) that in these cases agent $J+1$ follows the majority. Therefore it is enough to show that agent $J+2$ necessarily mimics the action of agent $J+1$. This is clear when $s_{J+2} = a_{J+1}$, namely the signal that agent $J+2$ receives is not in contradiction with the action taken by agent $J+1$. Thus, it remains to show that even when $s_{J+2} \neq a_{J+1}$ agent $J+2$ will choose the action a_{J+1} . Without loss of generality (WLOG) it is enough to prove this for the case $a_{J+1} = 1$ and $s_{J+2} = 0$. Formally, we need to prove that $Prob(\theta = 1 | a_{J+1} = 1 \cap S_{J+2} = 0) > 0.5$ whenever $J \geq \frac{1}{\delta^2 \epsilon}$.

$$\text{By Bayes rule } Prob(\theta = 1 | a_{J+1} = 1 \cap S_{J+2} = 0) = \frac{Prob(\theta=1)Prob(a_{J+1}=1 \cap S_{J+2}=0 | \theta=1)}{\sum_{j=0}^1 Prob(\theta=j)Prob(a_{J+1}=1 \cap S_{J+2}=0 | \theta=j)}.$$

Note that conditional on θ the random variables a_{J+1} and S_{J+2} are independent. In addition, $Prob(S_{J+2} = 0 | \theta = 1) = 0.5 - \delta$, $Prob(S_{J+2} = 0 | \theta = 0) = 0.5 + \delta$ and from lemma 3.2 and observation 3.1 $Prob(a_{J+1} = 1 | \theta = 1) \geq 1 - \frac{\epsilon}{4}$ and $Prob(a_{J+1} = 1 | \theta = 0) \leq \frac{\epsilon}{4}$

whenever $J \geq \frac{1}{\delta^2 \epsilon}$. Together with the trivial inequality $Prob(a_{J+1} = 1 | \theta = 1) \leq 1$ and since $\epsilon \leq (0.5 - \delta)$ we can conclude that:

$$\begin{aligned} Prob(\theta = 1 | a_{J+1} = 1 \cap S_{J+2} = 0) &\geq \\ \frac{0.5(1 - \frac{\epsilon}{4})(0.5 - \delta)}{0.5(1)(0.5 - \delta) + 0.5\frac{\epsilon}{4}(0.5 + \delta)} &\geq \\ \frac{(1 - \frac{\epsilon}{4})(0.5 - \delta)}{(0.5 - \delta) + \frac{(0.5 - \delta)}{4}(1)} = \frac{(1 - \frac{\epsilon}{4})}{\frac{5}{4}} = 0.8 - \frac{\epsilon}{5} &> 0.5 \end{aligned}$$

Q.E.D

Theorem 3.7. *For any $0 < \epsilon$ there exist \hat{N} such that for any $N > \hat{N}$ the social network \mathcal{H} is ϵ -optimal .*

Proof: WLOG we can assume that $\epsilon \leq 0.5 - \delta$.

Note that in the proof we use lemmas 3.4, 3.5, 3.6 and observation 3.1 switching ϵ by $\frac{\epsilon}{4}$.

Let $J = \frac{1}{\delta^2 \epsilon}$, $K = \frac{8}{\epsilon} \ln(\frac{4}{\epsilon})$ and $\hat{N} = \frac{128}{\epsilon^3 \delta^2} \ln \frac{4}{\epsilon} = 2 \frac{8JK}{\epsilon} \geq \frac{8JK}{\epsilon} + J$.

Let A be the event that among the first J agents that arrived there is at least one agent from the set K and let B be the event that among the agents that arrived in places $J + 1$ to $N \frac{\epsilon}{4}$ there is no agent from the set K .

By Lemma 3.4 $P(A) \leq \frac{\epsilon}{4}$ which implies $P(\bar{A}) \geq 1 - \frac{\epsilon}{4}$. By Lemma 3.5 $P(\bar{B} | \bar{A}) \geq (1 - \frac{\epsilon}{4})$ and so $P(\bar{B} \cap \bar{A}) \geq (1 - \frac{\epsilon}{4})^2$.

$E(X) = P(A)(E(X)|A) + P(\bar{A})[P(B|\bar{A})(E(X)|B \cap \bar{A}) + P(\bar{B}|\bar{A})(E(X)|\bar{B} \cap \bar{A})]$ which implies that $E(X) \geq P(\bar{B} \cap \bar{A})E(X|\bar{B} \cap \bar{A}) \geq (1 - \frac{\epsilon}{4})^2 E(X|\bar{B} \cap \bar{A})$.

Assume we can show that $E(X|\bar{B} \cap \bar{A}) \geq (1 - \frac{\epsilon}{4})(N - N \frac{\epsilon}{4})$ then we will be able to conclude that $E(X) \geq (1 - \frac{\epsilon}{4})^4 N \geq (1 - \epsilon)N$ as required.

To finish the proof we now show that indeed $E(X|\bar{B} \cap \bar{A}) \geq (1 - \frac{\epsilon}{4})(N - N \frac{\epsilon}{4})$.

The event \bar{A} implies that the initial set of $J = \frac{1}{\delta^2 \epsilon}$ were all non-celebrities and hence observe no predecessors. This means that they are all guinea-pigs. Since \bar{B} occurred the first celebrity arrived prior to time $N \frac{\epsilon}{4}$ agents. This agent observed more than J guinea-pigs. By Lemma 3.2 a strict majority of these agents took the optimal action (their action equal their signal) with probability greater than $1 - \frac{\epsilon}{4}$. Therefore by observation 3.1 the first celebrity followed this action. Any agent joining after stage time $N \frac{\epsilon}{4}$ is either a non-celebrity in which case he follows the action of the first celebrity (follows from lemma 3.6) or is himself a celebrity in which case he sees the same strict majority of initial J celebrities which he mimics. We conclude that conditional on the event $\bar{B} \cap \bar{A}$ all agents arriving after time $N \frac{\epsilon}{4}$ take an optimal action with probability $1 - \frac{\epsilon}{4}$ and our claim follows.

Q.E.D

3.3 An adversarial arrival order

We now consider an adversarial setting regarding the order at which agents arrive. The ϵ -asymptotic design will require randomness on the part of the designer. In particular we pursue a random celebrities graph where the set of celebrities, K , is not necessarily the set of agents $1, \dots, K$ but is rather chosen randomly. The way we choose a random celebrities graph is by randomly relabeling the agents and then applying the standard celebrities graph to the new labeling - agents whose labels are $1, \dots, K$ are designated as celebrities and those labeled $K + 1, \dots, N$ are designated as commoners.

More formally, let $\tau : N \rightarrow N$ be a random permutation of the names of the agents used by the designer (this is different from the order they arrive). In particular let τ be similar to σ_U in that it chooses each permutation with equal probability. Let \mathcal{H} be the (random) celebrities graph where agent k is a celebrity if and only if $\tau(k) \leq K$.

Before considering the case of random relabeling let us study the case where the relabeling is deterministic:

Lemma 3.8. *Let τ and σ be two deterministic permutations of the N agents and consider the following two social learning challenges which differ on the order of arrival and corresponding social networks*

1. *Agents arrive according to the order σ with signals $s_{\tau(1)}, s_{\tau(2)}, \dots, s_{\tau(N)}$ and \mathcal{H} is the bi-clique where agents $\tau^{-1}(1), \dots, \tau^{-1}(K)$ are designated as celebrities.*
2. *Agents arrive according to the permutation $\sigma(\tau^{-1})$ with signals s_1, s_2, \dots, s_N and \mathcal{G} is the bi-clique with agents $1, \dots, K$ designated as celebrities.*

For any signal profile s_1, s_2, \dots, s_N and for any stage t , the signal, the action and the location (celebrity or not) of the agent arriving at time t in both challenges are the same.

Proof:

We prove this by induction. For $t = 1$ the first agent to arrive in \mathcal{H} is $\sigma^{-1}(1)$ and he has the signal $s_{\tau(\sigma^{-1}(1))}$. On the other hand the first agent to arrive in \mathcal{G} is $\tau(\sigma^{-1}(1))$ which has the same signal $s_{\tau(\sigma^{-1}(1))}$. Both have no additional information and so both also take the same action. Also, in \mathcal{H} agent $\sigma^{-1}(1)$ is a celebrity if and only if $\tau(\sigma^{-1}(1)) \leq K$ which is exactly the condition for agent $\tau(\sigma^{-1}(1))$, the one arriving at time $t = 1$ in \mathcal{G} , to be a celebrity.

Now assume the induction hypothesis holds for all agents arriving at times $1, \dots, t$. Similar arguments as those made in $t = 1$ show that the agents arriving at time $t + 1$ in both networks have the same signal and are on the same side of the bi-partitive graph. By the induction hypothesis they also see the network spanned by the agents arriving at times $1, \dots, t$ are isomorphic (where the isomorphism function of agents is by the time they arrive). In addition, the isomorphism saves the action profile. Hence the agents arriving at time t have the same additional information and so once again must take a similar action.

Q.E.D

Note that Lemma 3.8 is false if we were to assume that the signal vector in both networks is the same (e.g., had we replaced $s_{\tau(1)}, s_{\tau(2)}, \dots, s_{\tau(N)}$ with s_1, s_2, \dots, s_N in \mathcal{H}). However, as

we now turn to show, whenever the signal vector is random, as in our model, the following holds.

Lemma 3.9. *Let τ and σ be two deterministic permutations of the N agents and consider the following two social learning challenges which differ on the order of arrival and corresponding social networks*

1. *Agents arrive according to the order σ and \mathcal{H} is the bi-clique where agents $\tau^{-1}(1), \dots, \tau^{-1}(K)$ are designated as celebrities.*
2. *Agents arrive according to the permutation $\sigma(\tau^{-1})$ and \mathcal{G} is the bi-clique with agents $1, \dots, K$ designated as celebrities.*

The expected number of agents taking the optimal action is the same, where the expectation is taken with respect to the random signal.

Proof:

Assume agents in \mathcal{H} receive the signal vector s while those in \mathcal{G} receive the signal vector $\tau(s)$. From Lemma 3.8 we conclude that the expected number of agents to take an optimal action in both cases is equal. In addition note that as any signal vector s has the same probability as the signal vector $\tau(s)$ (this is the exchangeability property of conditionally independent signals) and as the function $s \rightarrow \tau(s)$ is one-to-one and onto the distribution over action profiles in \mathcal{G} when agents have the receive the signal s and when agents receive the signal $\tau(s)$ is the same. The claim now follows.

Q.E.D

As the above holds for any deterministic permutations it must also hold for random permutations. Therefore we now have:

Corollary 3.10. *Let τ and σ be two random permutations of the N agents and consider the following two social learning challenges which differ on the order of arrival and corresponding social networks*

1. *Agents arrive according to the order σ and \mathcal{H} is the bi-clique where agents $\tau^{-1}(1), \dots, \tau^{-1}(K)$ are designated as celebrities.*
2. *Agents arrive according to the permutation $\sigma(\tau^{-1})$ and \mathcal{G} is the bi-clique with agents $1, \dots, K$ designated as celebrities.*

The expected number of agents taking the optimal action is the same, where the expectation is taken with respect to the random signal and the two permutations, σ and τ .

Let us consider the special case where τ is chosen from the uniform distribution over the set of all permutations. In this case we observe that the random permutation $\sigma(\tau^{-1})$ is also uniform over all permutations. This implies that whenever agents arrive according to the permutation $\sigma(\tau^{-1})$ and \mathcal{G} is the bi-clique with agents $1, \dots, K$ designated as celebrities we are back to the case of Theorem 3.7.

We are now ready to prove the existence of a random celebrities graph that is ϵ -optimal for a large enough society. Given $\epsilon > 0$ and a set of N agents let \mathcal{H} denote the random bi-clique where τ is chosen uniformly over all permutations and agent n is a celebrity if and only if $\tau(n) \leq K = \frac{8}{\epsilon} \ln \frac{4}{\epsilon}$. Then:

Theorem 3.11. *For any $0 < \epsilon$ there exists \hat{N} , such that for any $N > \hat{N}$ the random network \mathcal{H} is ϵ -optimal for any arrival order τ .*

Proof:

WLOG assume $\epsilon \leq (0.5 - \delta)$. As the permutation $\sigma(\tau^{-1})$ chooses all permutations with equal probability the claim follows from theorem 3.7 and corollary 3.10.

Q.E.D

3.4 General random arrival order

In the previous section we applied the celebrities network in a random fashion to provide a social network that works well for any (adversarial) arrival order. However, let us consider, once again, the Bayesian setting where the order of arrival is unknown but the distribution over arrival orders is known. In contrast with section 3.2 we will not assume a uniform random order but an arbitrary one, denoted σ .

Theorem 3.12. *For any $0 < \epsilon$ there exist \hat{N} such that for any $N > \hat{N}$ and any arrival order σ there exists some (deterministic) social network that is ϵ -optimal .*

Proof:

By Theorem 3.11 we know that for $\epsilon > 0$ there exist \hat{N} such that for any $N > \hat{N}$ there exists some random social network that is ϵ -optimal for any arrival order σ and in particular for a random arrival order σ . Recall the definition of an ϵ -optimal network is that the expected ratio of the number of agents that take an optimal action is larger than $1 - \epsilon$. However, this expectation can be computed as the expectation over the conditional expectation when conditioning on the realized network. Thus, as the expectation of the conditional expectation is larger than $1 - \epsilon$ it must be the case that for some realization this holds. as required.³

Q.E.D

4 Discussion

In this paper we dealt with the construction of social networks that will allow for efficient learning, yielding almost optimal behavior with overwhelming probability. This work is the first to show such off-line design, dealing with random arrival of agents.

Our results will also hold when we extend the model beyond the binary and beyond the symmetric case, although the exact size of the required population, as a function of the required proximity to optimality may change.

³This result is in the spirit of the so-called probabilistic method [2].

Social structures where a small set of celebrities interact with many commoners, and where information, once aggregated/received by a celebrity, propagates to the overall set of commoners, can be easily detected in reality. Network design is already a practice used by major social networks who strongly effect which peer information one sees and which is filtered as well as which ‘friends’ we follow more frequently. Combining network design with spontaneous network evolution and the proper incentives required for effective social learning goes beyond the scope of this work. We see this as a promising research direction.

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